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A MINIMAX THEOREM FOR IRREDUCIBLE COMPACT OPERATORS IN L^p-SPACES

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Dedicated to H. G. Tillmann on his 60th birthday

ABSTRACT

Let (X, Σ, μ) be a σ-finite measure space, T a compact irreducible (positive, linear) operator on $L^p(\mu)$ $(1 \leqq p < +\infty)$. It is shown that the spectral radius r of T is characterized by the minimax property

$$\sup_{f \in Q} \inf_{E \in \Sigma_0} \frac{\int_E Tf d\mu}{\int_E f d\mu} = r = \inf_{f \in Q} \sup_{E \in \Sigma_0} \frac{\int_E Tf d\mu}{\int_E f d\mu}$$

where Σ_0 denotes the ring of sets of finite measure and where Q denotes the set of all, almost everywhere positive functions in L^p. Moreover, if $r > 0$ then equality on either side is assumed iff f is the (essentially unique) positive eigenfunction of T. Various refinements are given in terms of corresponding relations for irreducible finite rank operators approximating T.

The present paper was inspired by a minimax theorem, due to Wielandt [6], for irreducible (non-negative) $(n \times n)$-matrices (see Corollary of Theorem 1). While Wielandt's result is quite easy to prove using functional analytic methods, its extension to compact operators on L^p-spaces leaves room for a deeper analysis.

1. Preliminaries

We consider (linear) positive operators on a (real or complex) space $L^p(X, \Sigma, \mu)$, where $1 \leqq p \leqq +\infty$ and (X, Σ, μ) is a σ-finite measure space. Recall that an operator $T : L^p \to L^p$ is called *positive* if $f \geqq 0$ (i.e., $f(t) \geqq 0$ a.e.) implies $Tf \geqq 0$; in symbols, $T \geqq 0$. An operator $T \geqq 0$ is called *irreducible* if no closed

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lattice ideal $\neq \{0\}$, L^p is invariant under T ([5]); if $p < +\infty$, this is equivalent to requiring that whenever $E \in \Sigma$ and $f(t) = 0$ a.e. on E implies $Tf(t) = 0$ a.e. on E, then $\mu(E) = 0$ or $\mu(X \setminus E) = 0$. Cf. [5, p. 157/58].

In case $p = +\infty$, we will restrict attention to operators T which are weak* continuous (equivalently, order continuous); the condition just given is then equivalent to irreducibility of the pre-adjoint of T on $L^1(\mu)$. By the spectral radius $r(T)$ of T we understand, in case of a real $L^p(\mu)$, the spectral radius of the canonical extension of T to the complexification of $L^p(\mu)$.

For the convenience of the reader, let us recall the following two well-known results that will be important in the sequel.

LEMMA 1. *If $T: L^p \to L^p$ is irreducible and compact and if $r = r(T) > 0$, then r is an eigenvalue of T of algebraic and geometric multiplicity* 1 *and the eigenspace of T is spanned by a function f such that $f(t) > 0$* a.e. (μ).

A proof can be found, for example, in [5, p. 329].

LEMMA 2. *Let $T \geqq 0$ be an operator on L^p, and let $\lambda \in \mathbf{R}$ be in the resolvent set of T. If $(\lambda - T)^{-1}$ is a positive operator, then $\lambda > r(T)$.*

PROOF. It is well known that for $T \geqq 0$, $r(T)$ is an element of the spectrum of T [5, V.4.1]. Now suppose that $\lambda < r(T)$ and $(\lambda - T)^{-1} = R(\lambda, T) \geqq 0$. If $\mu > r(T)$, then $R(\mu, T) \geqq 0$ (from the C. Neumann's series) and by the resolvent equation, we have

$$R(\mu, T) = R(\lambda, T) + (\lambda - \mu) R(\mu, T) R(\lambda, T) \leqq R(\lambda, T)$$

for all $\mu > r(T)$. But this implies $\| R(\mu, T) \| \leqq \| R(\lambda, T) \|$ for all $\mu > r(T)$ and hence that $R(\mu, T)$ is bounded as $\mu \downarrow r(T)$, a contradiction. □

REMARK. The above result is actually valid for positive operators on arbitrary complex Banach lattices (and certain ordered B-spaces) without assuming λ to be real, but we will have no need for that generality.

2. A minimax theorem

Let T be a positive operator on $L^p(\mu)$, let Q denote the set of all $f \in L^p(\mu)$ such that $f(t) > 0$ a.e. (μ), and let Σ_0 denote a subfamily of Σ such that $\mu(E) < +\infty$ for each $E \in \Sigma_0$ and the set $\{\chi_E : E \in \Sigma_0\}$ is weakly total in $L^q(\mu)$ (weak* total if $q = +\infty$), where $p^{-1} + q^{-1} = 1$.

Finally, for each $f \in Q$ let

$$(1) \qquad \rho(f) = \inf_{E \in \Sigma_0} \frac{\int_E Tf d\mu}{\int_E f d\mu}, \qquad \sigma(f) = \sup_{E \in \Sigma_0} \frac{\int_E Tf d\mu}{\int_E f d\mu}.$$

As before, by $r = r(T)$ we denote the spectral radius of T.

LEMMA 3. *For each $f \in Q$, we have $\rho(f) \leqq r$.*

PROOF. Suppose that $\rho(f) \geqq r + \varepsilon$ for some $f \in Q$ and $\varepsilon > 0$.

Writing (f, g) for the standard bilinear form placing $L^p(\mu)$ and $L^q(\mu)$ in duality, from (1) we obtain $(Tf, \chi_E) \geqq (r + \varepsilon)(f, \chi_E)$ for all $E \in \Sigma_0$. By the hypothesis on Σ_0 this implies $Tf \geqq (r + \varepsilon)f$. Successive application of T ($\geqq 0$) to this inequality leads to $T^n f \geqq (r + \varepsilon)^n f \cdot$ which, in turn, yields

$$\| T^n \| \geqq (r + \varepsilon)^n$$

for all $n \in \mathbf{N}$. The familiar conclusion is now that the spectral radius of T is $\geqq r + \varepsilon$, a contradiction. □

LEMMA 4. *If $p < +\infty$ (or $p = +\infty$ and T is weak* continuous) and if the spectrum of T is nowhere dense with respect to $\mathbf{R}_+$, then for each $f \in Q$ we have $\sigma(f) \geqq r$.*

PROOF. Suppose that $\sigma(f) \leqq r - \varepsilon$ for some $f \in Q$ and $\varepsilon > 0$. (Obviously we can suppose that $r > 0$ and $0 < \varepsilon < r$.) Then, as in the proof of Lemma 3, we obtain $Tf \leqq (r - \varepsilon)f$. Now the principal lattice ideal E_f of L^p generated by f is invariant under T, and isometrically isomorphic (as a Banach lattice with unit ball the order interval $[-f, f]$) to $C(K)$ for suitable compact K (Kakutani's Theorem, [5, II.7.4]). The positive operator T_0 induced by T on E_f thus has norm $\leqq r - \varepsilon$ and, consequently, spectral radius $\leqq r - \varepsilon$; in particular $(\lambda - T_0)^{-1}$ exists and is $\geqq 0$ whenever $\lambda > r - \varepsilon$. By hypothesis, there exists a λ, $r - \varepsilon < \lambda < r$, for which $(\lambda - T)^{-1}$ exists. Since E_f is dense (weak* dense if $p = +\infty$) in $L^p(\mu)$, $(\lambda - T)^{-1}$ must be the continuous (weak* continuous if $p = +\infty$) extension of $(\lambda - T_0)^{-1}$. But the latter is a positive operator and hence $(\lambda - T)^{-1}$ must be positive, which contradicts Lemma 2. Therefore, $\sigma(f) \geqq r$. □

THEOREM 1. *Let $T \geqq 0$ be a compact operator on $L^p(\mu)$, where $1 \leqq p \leqq +\infty$ and (X, Σ, μ) is σ-finite. If $p < +\infty$, assume T to be irreducible; if $p = +\infty$, assume T to be the adjoint of an irreducible operator on $L^1(\mu)$.*

The spectral radius r of T satisfies

$$\sup_{f \in Q} \rho(f) = r = \inf_{f \in Q} \sigma(f).$$

Moreover, if $r > 0$ either extremum is attained for some $f \in Q$ iff f is the (essentially unique) positive eigenfunction of T.

REMARK. It is known that $r > 0$ whenever ($\dim L^p(\mu) \geqq 2$ and) $p = 1$ or $p = +\infty$, or when μ has an atom [4], [5, V.6]; if it does occur that $r = 0$ (cf. Sec. 5) then $\rho(f) = 0$ for every, and $\sigma(f) = 0$ for no $f \in Q$.

PROOF OF THEOREM 1. By lemmata 3 and 4, we have $\rho(f) \leqq r \leqq \sigma(f)$ for each $f \in Q$. Suppose first that $r = 0$; we have to prove that $\inf \sigma(f) = 0$. Now for every $\varepsilon > 0$, the resolvent $R(\varepsilon, T)$ satisfies $TR(\varepsilon, T) \leqq \varepsilon R(\varepsilon, T)$ [5, p. 323]; choosing $g > 0$ arbitrarily and letting $f = R(\varepsilon, T)g$ we obtain $Tf \leqq \varepsilon f$. Since the principal lattice ideal generated by f is T-invariant it must be dense in $L^p(\mu)$ (weak* dense if $p = +\infty$). Hence we have $f \in Q$ and, obviously, $\sigma(f) \leqq \varepsilon$.

We henceforth suppose that $r > 0$. Since T is irreducible and compact, by Lemma 1 there exist eigenfunctions f_0, φ_0 satisfying $rf_0 = Tf_0$ and $r\varphi_0 = T'\varphi_0$ (where T' denotes the adjoint of T if $p < +\infty$, the pre-adjoint if $p = +\infty$) which are > 0 a.e. (μ). It is clear that $\rho(f_0) = r = \sigma(f_0)$. Conversely, if $\rho(f) = r$ for some $f \in Q$ then $Tf \geqq rf$, and $0 \leqq (Tf - rf, \varphi_0) = 0$ implies that $Tf = rf$. Similarly, $\sigma(f) = r$ implies that $Tf = rf$. □

In case the measure algebra (Σ, μ) is finite (i.e., in case $L^p(\mu)$ is essentially $\mathbf{R}^n$ or $\mathbf{C}^n$ with an appropriate norm), the characterization of the spectral radius given in Theorem 1 is due to Wielandt [6] who based on it new proofs and extensions of the well known theorems of Perron and Frobenius (see also [5], Chap. I, Exerc. 16). Since this finite-dimensional version is closely related to the subsequent discussion we note it here as a corollary of Theorem 1, with obvious notation.

COROLLARY. *Let A be a non-negative, irreducible $(n \times n)$-matrix $(n \geqq 2)$. The spectral radius (i.e., the largest positive eigenvalue) of A is characterized by the extremal properties*

$$\sup_{x \gg 0} \inf_i \frac{(Ax)_i}{x_i} = r = \inf_{x \gg 0} \sup_i \frac{(Ax)_i}{x_i}.$$

Either extremum is attained for a vector $x \gg 0$ iff $rx = Ax$.

3. A convergence theorem

From a practical point of view, Theorem 1 of the preceding section has at least two disadvantages: First, a calculation of the numbers $\rho(f)$, $\sigma(f)$, for any given f, requires the evaluation of many quotients $(Tf, \chi_E)/(f, \chi_E)$; second, the theorem gives no information on the approximation of r by $\rho(f)$, $\sigma(f)$ except when f happens to be an eigenfunction of T. For a refined analysis, given in Sec. 3 below, we need a result on the behavior of the spectral radii and eigenfunctions of certain finite rank operators approximating T.

From now on we assume, without serious loss of generality, (X, Σ, μ) to be a *finite* measure space. Recall that the closed vector sublattices H of $L^p(\mu)$ $(1 \leqq p < +\infty)$, containing the constant functions, are in one-to-one correspondence with the (μ-complete) σ-subalgebras Σ' of Σ [5, III. 11.2], by virtue of $f \in H$ iff f is measurable Σ'; each such H is the range of a contractive positive projection $P : L^p(\mu) \to H$ which defines the conditional expectation with respect to Σ'. (For $p = 2$, P is the orthogonal projection of $L^2(\mu)$ onto H.)

LEMMA 5. *Let P be the conditional expectation projection on $L^p(\mu)$ $(1 \leqq p \leqq +\infty)$ with respect to some σ-subalgebra of Σ. If $T(\geqq 0)$ on $L^p(\mu)$ $(1 \leqq p < +\infty)$ is irreducible, then so is PT.*

PROOF. If $f \in L^p(\mu)$ then Pf is the unique Σ'-measurable function g such that $\int_A g d\mu = \int_A f d\mu$ for every $A \in \Sigma'$ [5, p. 211]. Suppose $T \geqq 0$ is irreducible and that $J = \{f : f(t) = 0 \text{ a.e. on } E\}$, where $E \in \Sigma$, is an ideal invariant under PT. Let $F = X \setminus E$ and $PT\chi_F = g$; then g must vanish a.e. (μ) on E. Let $N = \{t \in X : g(t) = 0\}$, then $N \in \Sigma'$ and $N \supset E$ (except for a μ-null set, of course). Now $0 = \int_N g d\mu = \int_N T\chi_F d\mu$ shows that $T\chi_F$ vanishes on N and, *a fortiori*, on E; this implies, however, that $T(J) \subset J$. Therefore, $J = \{0\}$ or $J = L^p$, and it follows that PT is irreducible. □

REMARK. From the preceding it follows that, if $T \geqq 0$ is an operator on $L^\infty(\mu)$ possessing no non-trivial, weak* closed invariant ideals, then neither does PT (and, in fact, PT is irreducible whenever P has finite rank).

In particular, if Σ'_α is the σ-subalgebra of Σ generated by a finite Σ-measurable partition α of X, then the associated projection is given by

$$P_\alpha = \sum_{E \in \alpha} \mu(E)^{-1} \chi_E \otimes \chi_E.$$

The set of all finite Σ-measurable partitions of X is directed under the relation $\leqq$ defined by "$\alpha_1 \leqq \alpha_2$ iff $\Sigma'_{\alpha_1} \subset \Sigma'_{\alpha_2}$". Now if A is any directed family of finite Σ-measurable partitions of X and if $\beta = \bigcup\{\alpha : \alpha \in A\}$, then the projection P_β associated with the σ-subalgebra Σ'_β generated by β, satisfies $P_\beta = \lim_{\alpha \in A} P_\alpha$ in the strong operator topology of $L^p(\mu)$ $(1 \leqq p < +\infty)$. (This is even true in $L^\infty(\mu)$ provided that β is already a σ-subalgebra of Σ.)

We now let $T_\alpha = P_\alpha T$, $T_\beta = P_\beta T$ and denote, as before, by $r(T)$ the spectral radius of an operator T.

THEOREM 2. *Let A be any directed family of finite, measurable partitions of X, and let $\beta = \bigcup\{\alpha : \alpha \in A\}$. If $T \geqq 0$ is any compact operator on $L^p(\mu)$ $(1 \leqq p < +\infty)$, then*

$$\lim_{\alpha \in A} r(T_\alpha) = r(T_\beta).$$

If, in addition, T is irreducible and if $r(T_\beta) > 0$, then

$$\lim_{\alpha \in A} f_\alpha = f_\beta$$

where $f_\alpha(f_\beta)$ denotes the unique, normalized positive eigenfunction of $T_\alpha(T_\beta)$ pertaining to $r(T_\alpha)(r(T_\beta))$.

PROOF. Note first that by compactness of T, we have $\lim_\alpha \| T_\alpha - T_\beta \| = 0$. Suppose first that $\lim_\alpha r(T_\alpha) = 0$. If we had $r(T_\beta) > 0$, we would have $r(T_\alpha) \leqq \frac{1}{2} r(T_\beta)$ for $\alpha \geqq \alpha_0$, say. Let λ denote a real number satisfying $\frac{1}{2} r(T_\beta) < \lambda < r(T_\beta)$, not contained in the spectrum of T_β. (Such numbers exist because T_β is compact.) Now $(\lambda - T_\alpha)^{-1}$ $(\alpha \geqq \alpha_0)$ converges in norm to $(\lambda - T_\beta)^{-1}$; but $(\lambda - T_\alpha)^{-1} \geqq 0$, since $\lambda > r(T_\alpha)$, and hence $(\lambda - T_\beta)^{-1} \geqq 0$ which contradicts Lemma 2.

The preceding argument actually shows that for any cluster point s of the net $\{r(T_\alpha) : \alpha \in A\}$, we must have $s \geqq r(T_\beta)$. On the other hand, a standard argument shows that every cluster point $s > 0$ of the net $(r(T_\alpha))$ must be an eigenvalue of T_β, whence it follows that $s \leqq r(T_\beta)$. Therefore, this net has the unique cluster point $r(T_\beta)$ which proves the first assertion.

Now suppose that $T \geqq 0$ is compact irreducible, and that $r(T_\beta) > 0$. Then $r(T_\beta)$ is an eigenvalue of T_β. On the other hand, T_β and each T_α is irreducible (Lemma 5) and, by Lemma 1, $r(T_\beta)$ and all $r(T_\alpha)$ have total multiplicity 1. (For $r(T_\beta)$ this also follows from perturbation theory, [2, IV.3.5].) Thus if $f_\alpha \geqq 0$ is the unique normalized function satisfying $r(T_\alpha) f_\alpha = T_\alpha f_\alpha$, the net $\{f_\alpha : \alpha \in A\}$ has but one cluster point, namely, the unique normalized function $f_\beta \geqq 0$ satisfying $r(T_\beta) f_\beta = T_\beta f_\beta$. Since every subnet of (f_α) has a convergent subnet, it follows that $\lim_\alpha f_\alpha = f_\beta$. □

REMARK. The assertion of Theorem 2 carries over, as its proof shows, to arbitrary families of σ-subalgebras of Σ directed under inclusion. It also carries over to compact weak* irreducible operators T on $L^\infty(\mu)$, provided the net A is saturated with respect to the σ-algebra Σ'_β it generates, i.e. provided that $\Sigma'_\beta = \bigcup_{\alpha\in A} \Sigma'_\alpha$.

4. Refinements

We return to the characterization, given in Theorem 1, of the spectral radius r of a compact, irreducible operator on $L^p(X,\Sigma,\mu)$. We will assume that $1 \leqq p < +\infty$ and that (X,Σ,μ) is a separable, finite measure space.

An increasing sequence (α_n) of finite, measurable partitions of X (cf. Sec. 3) will be called *total* if the σ-subalgebra Σ'_β generated by $\beta = \bigcup_n \alpha_n$ (and completed by adding all μ-null sets) equals Σ.

Again, let Q denote the set $\{f \in L^p : f(t) > 0 \text{ a.e. } (\mu)\}$ and let (α_n) be total. For each $f \in Q$, we define

$$(1^*) \qquad \rho_n(f) = \inf_{E\in\alpha_n} \frac{\int_E Tf d\mu}{\int_E f d\mu}, \qquad \sigma_n(f) = \sup_{E\in\alpha_n} \frac{\int_E Tf d\mu}{\int_E f d\mu}.$$

To simplify notation, we write P_n for the projection P_{α_n} (Sec. 3) and let $T_n = P_nT$, $S_n = P_n(L^p)$, $r_n = r(T_n)$, $r = r(T)$. Moreover, we note that for each $f \in Q$, $(\rho_n(f))$ is a non-increasing and $(\sigma_n(f))$ a non-decreasing sequence in $\mathbf{R}$.

THEOREM 3. *Let $T \geqq 0$ be an irreducible compact operator with spectral radius r on $L^p(X,\Sigma,\mu)$, where $1 \leqq p < +\infty$ and (X,Σ,μ) is a separable finite measure space. Let (α_n) be any increasing total sequence of finite, measurable partitions of X and define ρ_n, σ_n as in (1*) above. The following assertions hold:*

(i) *For each $f \in Q$, $\lim_n \rho_n(f) \leqq r \leqq \lim_n \sigma_n(f)$.*

(ii) *Every sequence in Q converging to an eigenfunction of T contains a subsequence (g_k) such that for all n,*

$$\lim_{k\to\infty} \rho_n(g_k) = r = \lim_{k\to\infty} \sigma_n(g_k).$$

(iii) *Conversely, if (f_n) is a normalized sequence in Q such that $f_n \in S_n$ for all n and $\lim_n \rho_n(f_n) = r > 0$, then (f_n) converges to the unique normalized function $f \in Q$ satisfying $rf = Tf$.*

PROOF.

(i) By Theorem 2, we have $r(T_n) = r_n \to r$. On the other hand, $\rho_n(f) = (Tf,\chi_E)/(f,\chi_E)$ for some $E \in \alpha_n$. Thus $(Tf,\chi_E) = (T_nf,\chi_E)$ and $\rho_n(f) = (T_nf,\chi_E)/(f,\chi_E)$. By Theorem 1 (applied to T_n), $\rho_n(f) \leqq r_n$; therefore, $\lim_n \rho_n(f) \leqq r$. The proof of the second assertion contained in (i) is similar.

(ii) Let (f_k) denote a sequence in Q converging to an eigenfunction f of T; it follows that $f \geqq 0$ and hence, by a well known result on irreducible operators [5, V.5.2], that $r > 0$ and $rf = Tf$. Now let $n \in \mathbf{N}$ be fixed. Then for each $k \in \mathbf{N}$, there exists a set $E_k \in \alpha_n$ such that $\rho_n(f_k) = (Tf_k, \chi_{E_k})/(f_k, \chi_{E_k})$. Since α_n is finite, the same element of α_n must occur infinitely often among the sets E_k, say the set $E_0 \in \alpha_n$. Now let h_k denote a subsequence of f_k such that $\rho_n(h_k) = (Th_k, \chi_{E_0})/(h_k, \chi_{E_0})$. But $h_k \to f$ and so

$$\lim_{k \to \infty} \rho_n(h_k) = (Tf, \chi_{E_0})/(f, \chi_{E_0}) = r.$$

Similarly, $\lim_{k\to\infty} \sigma_n(h'_k) = r$ for a suitable subsequence of (h_k). A standard diagonal argument now completes the proof of (ii).

(iii) We have to treat the cases $p = 1$ and $p > 1$ separately.

Case $p > 1$: We choose n_0 so that for $n \geqq n_0$, $\rho_n(f_n) \geqq r - \varepsilon$ where $\varepsilon > 0$ is preassigned but satisfies $0 < \varepsilon < r$. By (1*), we have $(Tf_n, \chi_E) \geqq (r - \varepsilon)(f_n, \chi_E)$ for all $E \in \alpha_n$. Since $f_n \in S_n$, it follows that

$$(*) \qquad T_n f_n \geqq (r - \varepsilon) f_n \qquad (n \geqq n_0).$$

In particular, we have $\| T_n f_n \| \geqq r - \varepsilon$ for $n \geqq n_0$, since $\| f_n \| = 1$.

Now let h be a weak cluster point of (f_n), then Th is a strong cluster point of (Tf_n); by (*) we have $\| Th \| \geqq r$, so $h \neq 0$. By Eberlein's theorem, there exists a subsequence (g_n) of (f_n) converging to h weakly; then $Tg_n \to Th$ in norm and we obtain $r \| h \| = \| Th \| \geqq r$, that is, $\| h \| = 1$. From the uniform convexity of L^p $(1 < p < +\infty)$ it now follows that $g_n \to h$ in norm, and $h = f$ is the unique, normalized positive eigenfunction of T. We have shown that every weak cluster point of (f_n) is a strong cluster point, and necessarily equal to f. This implies $\lim_n f_n = f$.

Case $p = 1$: With the same initial argument as in case $p > 1$, we arrive at

$$(**) \qquad T_n f_n \geqq (r - \varepsilon_n) f_n \qquad (n \geqq n_0)$$

where now $\varepsilon_n = (r - \rho_n(f_n))^+$. There exists a convergent subsequence of $(T_n f_n)$; in fact we can assume that

$$\| T_{n(k+1)} f_{n(k+1)} - T_{n(k)} f_{n(k)} \| \leqq 2^{-(k+1)}$$

for a suitable sequence $n(k)$ of integers. Let

$$h = T_{n(0)} f_{n(0)} + \sum_{k=0}^{\infty} | T_{n(k+1)} f_{n(k+1)} - T_{n(k)} f_{n(k)} |.$$

Then we have $(r - \varepsilon_{n(k)}) f_{n(k)} \leqq h$. Since the order interval $[0, h]$ in $L^1(\mu)$ is weakly compact, it follows that (f_n) has a weakly convergent subsequence. For its weak limit g we obtain from (**): $Tg \geqq rg$, which in the familiar fashion implies $Tg = rg$. Also $\| g \| = 1$, since $\| f_n \| = 1$ for all n and the norm of $L^1(\mu)$ is weakly continuous on the positive cone of $L^1(\mu)$. Also, as before, $g = f$ where f is the unique normalized function $\geqq 0$ satisfying $rf = Tf$.

We have shown that each subsequence of (f_n) contains a subsequence weakly convergent to f; it follows that $f_n \to f$ weakly. Consequently, $T_n f_n \to Tf$ in norm, since T is compact and $\| T_n - T \| \to 0$. Therefore, the sequence $(T_n f_n - (r - \varepsilon_n) f_n)$ converges weakly to 0; but its elements are $\geqq 0$ by (**), and hence this sequence norm converges to 0. Since $\varepsilon_n \to 0$, it finally follows that (f_n) norm converges to f.

□

5. Concluding remarks

At various points of this paper, the desired conclusions were dependent on the assumption that the spectral radius r of a compact irreducible operator be > 0. As pointed out after Theorem 1, the problem if necessarily $r > 0$ is open for $1 < p < +\infty$ and μ a diffuse measure, and it can be shown that $r = 0$ can occur only on some $L^p(\mu)$ if it occurs on $L^2(\mu)$ (for the same or an equivalent measure). On the other hand, it can also be shown that for a large class of irreducible compact operators on $L^p(\mu)$ (notably those of type l^r in the sense of Pietsch [3] or integral operators in the sense of Luxemburg–Zaanen [7]) the spectral radius is > 0, and similar results are valid for the spectral bound of irreducible one-parameter semi-groups.

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